THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. X. OCTOBER, 1903. No. 10.

ON THE GROUPS OF THE FIGURES OF ELEMENTARY GEOMETRY.

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Among the various elementary illustrations of the group concept those which relate to the movements of space are perhaps the most instructive. Even those students who have no trouble in approaching the subject by analytic methods generally take pleasure in observing the geometric interpretation of some of the most useful groups. In what follows we shall aim to determine the groups of the most common geometric figures rather than to make use of these figures for the sake of illustrating known groups. Only the most elementary notions about group theory are presupposed. Only groups of finite order are here considered.

The group of a figure is composed of all the movements of space which transform the figure into itself. In these movements space is regarded as rigid, that is, any two points are transformed into two points at the same distance from each other as the original points.

All the movements which transform a system of points into itself must also transform a certain point into itself. This elementary fact may be proved as follows: There is at least one minimum sphere which includes all these points. If there were two distinct minima spheres having this property, all the given points would be in the space common to the two spheres. This is impossible since this common space is in a smaller sphere; viz., in the one whose radius is equal to the radius of the circle which is composed of the points common to the two equal spheres. Every movement which transforms a system of points into itself must therefore transform the center of the minimum circumscribing

sphere into itself. Hence such movements may be composed of rotations around this center and ever concentric sphere must be transformed into itself.

We shall first consider the rotations which transform a plane triangle into itself. If the triangle is equilateral there are three such rotations around the axis perpendicular to its plane at the center; viz., the rotations through 0°, 120°, and 240°, respectively. These three rotations clearly form the group of order 3. This, however, is not the group of the triangle, for there are three other rotations which transform it into itself; viz., the rotations around the lines of symmetry through 180°. The group formed by these six rotations (the equilateral triangle group) is evidently simply isomorphic with the group formed by all the possible permutations of three things,—the symmetric group of degree 3. It may be observed that any two of these rotations of period two are equivalent to one of period three and that the rotations of period two are non commutative; i. e., the rotation of period three which is equivalent to two rotations of period two depends upon the order in which they are taken. Hence the equilateral triangle group is non-commutative. It will be found that all the groups of the regular plane polygons and of the regular solids are non-commutative. All the groups of the spherical polygons are commutative. In fact, they are always cyclic:

If a triangle is isosceles without being also regular, its group is clearly of order two; while the identity is the group of the scalene triangle. The cyclic group of order 3 is the group of the equilateral spherical triangle. All other spherical triangles evidently admit no rotation besides the identity. Hence we see that *the group of the plane equilateral triangle is of order 6, that of the spherical equilateral triangle is of order 3, that of the plane isosceles triangle is of order 2. The other triangles admit no rotations besides the identity.*

Before considering the groups of movements of the quadrangles it may be desirable to consider the subject from a somewhat more general standpoint. Either all or just half of the movements which transform into itself a system of co-planar points finite in number consist of rotations around an axis perpendicular to the plane through the center of the minimum enclosing circle. Among these rotations there must be a smallest one and all the others are composed of repetitions of this smallest rotation. Hence all such rotations form a cyclic group. If there is any additional rotation it must be of period two. If this rotation is followed by each of the rotations of the given cyclic subgroup, in order, we obtain just as many distinct rotations of order two as there are operations in this subgroup. A group which contains a cyclic subgroup of half its order and only operations of order two besides those in this subgroup is known as a dihedral rotation group. *All the groups of movements of systems of co-planar points, finite in number, must therefore be either cyclic or of the dihedral rotation type.* It is easy to see that the dihedral rotation group of order $2n$ is the group of the regular plane polygon of n sides. On this account the dihedral rotation groups may be called the regular polygon groups. If s is any operation in the cyclic subgroup of order n and if t is any other operation of the group it follows that $st=$ some t and hence $stst=1$. Hence $t^{-1}st=s^{-1}$; that is, t transforms every s into its

inverse. This result can also be seen geometrically and a dihedral rotation group may be defined by the fact that it is composed of a cyclic group of order n and n operations of order two transforming each operation of this cyclic subgroup into its inverse. The regular polygon group of order $2n$ is therefore non-commutative whenever n exceeds two.

It is now easy to determine all the quadrangle groups. The largest of these is the group of the square. This consists of the cyclic group of order 4 and four additional operations of order 2 corresponding to the rotations around its four lines of symmetry. Since each of these eight rotations, with the exception of the identity, permutes at least two of the vertices, the square group can be represented on four letters and hence it is simply isomorphic with the octic group.* It contains two non-cyclic subgroups of order 4. The transitive one is the group of a rectangle and the intransitive one is the group of a rhombus. The other subgroup of order 4 is cyclic and it is the group of the spherical quadrangle whose vertices determine a square. All the other quadrangles belong either to the group of period 2 or to the identity.

It is evident that a crossed quadrangle whose vertices coincide with those of a square or rectangle, belongs to the group of order four. Since crossed polygons are not generally studied in elementary geometry we shall exclude them from our consideration. There are two classes of plane convex quadrangles whose group is of order 2; viz., (1) Those whose diagonals intersect at right angles, only one of them being bisected, (2) Those whose unequal diagonals bisect each other obliquely, the general parallelograms. The latter are the only plane quadrangles which permit a non-identical movement without having any line of symmetry.

If a quadrangle is concave, its group can differ from the identity only when the concave vertex is at the center of the circle circumscribing the other three vertices and when these vertices determine an isosceles triangle. As the group of such a quadrangle is of order two, there are just three classes of plane quadrangles which have this group. A spherical quadrangle belongs to this group if the common perpendicular of the lines joining two pairs of vertices bisects each of these lines, and the vertices of the quadrangle do not determine a square. All the other quadrangles admit no movement besides the identity.

There are only two substitution groups on three letters. These are the groups of movements of the plane and the spherical equilateral triangles. There are seven groups on four letters. Of the latter, the group of order 2 is the group of movements of various quadrangles, the three groups of order 4 are the groups of movements of the rectangle, rhombus, and spherical quadrangle whose vertices determine a square, respectively; and the octic group is the group of movements of the square. As the other two possible substitution groups of degree 4 are neither cyclic nor of the dihedral rotation type they cannot be the groups of movements of any polygons. It is, however, easy to verify that they are the groups of movements of the regular tetrahedron and cube respectively. Hence

*Pierpont, Annals of Mathematics, Vol. 1, 1900, page 140.

each one of the seven possible substitution groups on four letters is a group of movements.

Comparatively few of the groups whose degrees exceed 4 are groups of movements. The most interesting of these is the icosaedron group, which is simply isomorphic with the alternating group on five letters, and is the group of movements of the regular icosaedron and also of the regular dodecaedron. The regular octaedron has the same group of movements as the cube; viz., the symmetric group on four letters, while the regular tetraedron admits only the rotations corresponding to the alternating group on four letters. The orders of these groups can be easily found by observing the number of possible rotations of the solid which transform a face into itself and multiplying this number by the number of faces.

If the base of a regular pyramid has n sides ($n \neq 3$) its group is the cyclic group of order n. This is also the case when $n=3$ and the pyramid is not a regular tetraedron. If the base of a regular prism has n sides ($n \neq 4$) its group is the dihedral rotation group of order $2n$. When $n=4$ and a lateral edge is not equal to a base edge the group of the regular prism is also the dihedral rotation group of order $2n$. The rectangular parallelopiped with three unequal edges belongs to the rectangle group and it is easy to construct pyramids which belong to the group of order two. Hence *all the possible groups of movements of polygons are also groups of movements of either the pyramids or the prisms.*

It has been observed above that solids may have three additional groups of movements. That is, there are three groups of rotation which are neither cyclic nor of the dihedral rotation type. The proof that there are no more than three such groups is not very difficult. It is only necessary to observe that each complete set of conjugate vertices must have the same group and must lie on a certain sphere, whenever the number of these vertices exceeds two. Hence we may confine our attention to any one complete set of conjugate vertices provided it includes more than two.*

If a complete set of conjugate vertices (which may consist of just one vertex) lies on a plane which does not pass through the center of the minimum circumscribed sphere, the group of the solid is cyclic. When this condition is not satisfied and a complete set of conjugate vertices lies on two parallel planes the group of the solid must be metacyclic.

In all other cases the points where the conjugate axes of rotations meet the surface of the sphere on which the complete set of conjugate vertices lies must form the vertices of a regular polyhedron.† As the number of the former points must exceed two, their group is identical with the group of the vertices. These results may be stated as follows: *If the group of movements of a system of points is neither cyclic nor of the dihedral rotation type, it must be the tetraedron, octahedron, or the icosaedron group.*

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*If each conjugate set is composed of only two vertices it is easy to see that the group must be either of order two or of order four.

†For a more complete proof see Jordan, *Annali di Matematica*, Vol. 2, 1868; also Klein, *Ueber das Ikosaeder*, 1884, page 21.